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Connes' Distance Function on One-Dimensional Lattices

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We show that there is an operator with a simple geometric significance which yields the ordinary geometry of an open and closed linear equidistant lattice via Connes' distance function. Some related aspects of distances on graphs are briefly discussed.

1. INTRODUCTION

The notion of the distance between two points of a space is at the very origin of geometry and physics. It should be clear, however, that the classical Euclidean distance no longer makes sense below a certain length scale. Rather, it has to be replaced by some quantized version (which still has to be defined and related to measuring devices). It is hoped that generalizations of the distance concept and, more generally, generalizations of geometric concepts will guide us toward a new physical theory. In this context a proposal by Connes (see Connes, 1994, and references therein) appears to be of particular interest.

According to Connes, the geodesic distance function

$$d(p, q) = \text{infimum of length of paths from } p \text{ to } q \tag{1}$$

on a Riemannian manifold M can be reformulated as

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$$d(p, q) = \sup\{\left|f(p) - f(q)\right|; f \in \mathcal{A}, \left\|[\mathfrak{D}, \hat{f}]\right\| \leq 1\} \tag{2}$$

where $\mathcal{A}$ is a (suitably restricted) algebra of functions on M represented as multiplication operators $\hat{f}$ on a Hilbert space $\mathcal{H}$, and $\mathfrak{D}$ is the Dirac operator. The latter formulation can also be applied to discrete spaces and even generalized to "noncommutative spaces." A suitable replacement for the operator $\mathfrak{D}$ has to be found, however. In particular, one would like to find an operator counterpart of certain (simple) geometries in order to gain an understanding of how to set up a physical theory (e.g., some version of mechanics) in terms of the new mathematics.

Bimonte *et al.* (1994) and Atzmon (1996) considered a one-dimensional lattice with the choice

$$(\mathfrak{D}_{\text{s.d.}}\Psi)_k = \frac{1}{2i}(\Psi_{k+1} - \Psi_{k-1}), \qquad k \in \mathbb{Z} \tag{3}$$

The distances calculated with this *symmetric difference* operator turned out to be given by

$$d(0, 2n - 1) = 2n, \qquad d(0, 2n) = 2\sqrt{n(n + 1)} \qquad (n \in \mathbb{N}) \tag{4}$$

which looks quite remote from the expected result for a linear equidistant lattice.

In the following two sections we show that there is another operator which actually produces the expected result, for an open as well as for a closed linear lattice. Section 4 contains some related comments.

2. THE OPEN LINEAR LATTICE

We consider a *finite* set of N points. Then $\mathcal{A}$ is the algebra of all complex functions on it. $f \in \mathcal{A}$ will be represented by

$$f \mapsto \hat{f} = \begin{pmatrix} f_1 & 0 & & & & 0 \\ 0 & \ddots & & & & \\ & & f_N & & & \\ & & & f_1 & & \\ & & & & \ddots & \\ 0 & & & & & f_N \end{pmatrix} \tag{5}$$

where $f_k = f(k)$ (numbering the lattice sites by $1, \ldots, N$). We choose the operator

$$\hat{\mathcal{D}}_N = \begin{pmatrix} 0 & \mathcal{D}_N^\dagger \\ \mathcal{D}_N & 0 \end{pmatrix} \tag{6}$$

on $\mathcal{H} = \mathbb{C}^{2N}$, where

$$\mathcal{D}_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & 1 \\ 0 & \cdots & \cdots & & 0 \end{pmatrix} \tag{7}$$

Then $(\mathcal{A}, \mathcal{H}, \hat{\mathcal{D}}_N)$ is a *spectral triple*, a basic structure in Connes' approach to noncommutative geometry (Connes, 1995) (see also, Connes, 1996, for a refinement). It is called *even* when there is a grading operator. In the case under consideration such an operator is given by

$$\gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{8}$$

It is self-adjoint and satisfies

$$\gamma^2 = \mathbf{1}, \qquad \gamma\hat{\mathcal{D}}_N = -\hat{\mathcal{D}}_N\gamma, \qquad \gamma\hat{f} = \hat{f}\gamma \tag{9}$$

Let us now turn to the calculation of the distance function. With a complex function f we associate a real function F via

$$F_1 = 0, \qquad F_{k+1} = F_k + \left|f_{k+1} - f_k\right|, \qquad k = 1, \ldots, N-1 \tag{10}$$

Then $\left|F_{k+1} - F_k\right| = \left|f_{k+1} - f_k\right|$ and

$$\left\|[\hat{\mathcal{D}}_N, \hat{f}]\psi\right\| = \left\|[\hat{\mathcal{D}}_N, \hat{F}]\psi\right\| \tag{11}$$

for all $\psi \in \mathbb{C}^{2N}$. Consequently, in calculating the supremum over all functions f in the definition of Connes' distance function, it is sufficient to consider only *real* functions. Then $Q_N = i[\hat{\mathcal{D}}_N, \hat{f}]$ is Hermitian and its norm is given by the maximal absolute value of its eigenvalues. Instead of Q_N, it is simpler to consider

$$Q_N Q_N^\dagger = \mathrm{diag}(0, (f_2 - f_1)^2, \ldots, (f_N - f_{N-1})^2, \\ (f_2 - f_1)^2, \ldots, (f_N - f_{N-1})^2, 0) \tag{12}$$

which is already diagonal. This implies

$$\left\|[\hat{\mathcal{D}}_N, \hat{f}]\right\| = \max\{\left|f_2 - f_1\right|, \ldots, \left|f_N - f_{N-1}\right|\} \tag{13}$$

from which we conclude that $d(k, l) = \left|k - l\right|$.

Fig. 1. An oriented linear lattice graph.

The choice (3) for $\mathfrak{D}$ was motivated by a simple discretization procedure (which is known to cause the problem of fermion doubling in lattice field theories). There is, however, no reason why this operator must yield the plain geometry of a linear equidistant lattice via Connes' distance function. There are many geometries which can be assigned to a discrete set and these should correspond to the choice of some operator $\hat{\mathfrak{D}}$. Now it is certainly of interest to know what distinguishes our choice (6). This is built from the operator $\mathfrak{D}$ in such a way that $\hat{\mathfrak{D}}$ is self-adjoint. Moreover, the construction guarantees that there is a grading operator. So we are left to understand the significance of $\mathfrak{D}$. This matrix is the adjacency matrix of the oriented linear lattice graph (see Fig. 1). This digraph plays a basic role in a formulation of lattice theories in the framework of noncommutative geometry (Dimakis and Müller-Hoissen, 1994, and references cited therein).

Remark. Instead of using $\hat{\mathfrak{D}}$ to define Connes' distance function, we may use directly $\mathfrak{D}$ (which, in general, is not symmetric) and no doubling in the representation of f. A simple calculation in the case treated above shows that

$$\left\| [\mathfrak{D}_N, f] \right\| = \left\| [\hat{\mathfrak{D}}_N, \hat{f}] \right\| \tag{14}$$

so that we obtain the same distances as before.

3. THE CLOSED LINEAR LATTICE

Connecting in addition the last with the first point in the oriented digraph in Fig. 1, we find that the adjacency matrix becomes

$$\mathfrak{D}_{Nc} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix} \tag{15}$$

For $\Psi = (\phi, \psi) \in \mathbb{C}^{2N}$ we find

$$\begin{aligned} \left\| [\hat{\mathfrak{D}}_{Nc}, \hat{f}]\Psi \right\|^2 &= \left\| [\mathfrak{D}_{Nc}, f]\psi \right\|^2 + \left\| [\mathfrak{D}^{\dagger}_{Nc}, f]\phi \right\|^2 \\ &= \sum_{k=1}^{N} \left| f_{k+1} - f_k \right|^2 \left(\left| \psi_{k+1} \right|^2 + \left| \phi_k \right|^2 \right) \end{aligned}$$

$$\geq \sum_{k=1}^{N} \left\| f_{k+1} - a \right| - \left| f_k - a \right\|^2 (\left| \psi_{k+1} \right|^2 + \left| \phi_k \right|^2)$$

$$= \left\| [\mathfrak{D}_{Nc}, \hat{F}] \Psi \right\|^2 \tag{16}$$

where $F_k = \left| f_k - a \right|$. Here and in the following, an index $N + 1$ has to be replaced by 1. Choosing $a = f_1$, we have $\left\| [\mathfrak{D}_{Nc}, \hat{F}] \right\| \leq \left\| [\mathfrak{D}_{Nc}, \hat{f}] \right\|$ and $F_k = \left| f_k - f_1 \right|$. It follows that

$$d(1, n) = \sup\{ \left| F_n \right|; F \text{ real}, F_1 = 0, \left\| [\mathfrak{D}_{Nc}, \hat{F}] \right\| \leq 1\} \tag{17}$$

The condition $\left\| [\mathfrak{D}_{Nc}, \hat{F}] \right\| \leq 1$ is equivalent to $\left| F_{k+1} - F_k \right| \leq 1$, $k = 1, \ldots, N$. Let $n - 1 \leq N - n + 1$. It is then possible to set the first $n - 1$ terms in the identity

$$\underbrace{(F_2 - F_1) + \cdots + (F_n - F_{n-1})}_{n\,-\,1 \text{ terms}} + \underbrace{(F_{n+1} - F_n) + \cdots + (F_1 - F_N)}_{N\,-\,n\,+\,1 \text{ terms}} = 0 \tag{18}$$

each separately to 1. Using the trivial identity

$$\left| F_n \right| = \left| (F_2 - F_1) + (F_3 - F_2) + \cdots + (F_n - F_{n-1}) \right| \tag{19}$$

we now find $d(1, n) = n - 1$. If $n - 1 > N - n + 1$, then each of the last $N - n + 1$ terms in (18) can be set to 1. Using

$$\left| F_n \right| = \left| (F_1 - F_N) + (F_N - F_{N-1}) + \cdots + (F_{n+1} - F_n) \right| \tag{20}$$

we find $d(1, n) = N - n + 1$.

4. SOME COMMENTS

For a linear lattice we have recovered the ordinary distances from the adjacency matrix (which acts as a shift operator on functions on the lattice) in Connes' framework of noncommutative geometry. Corresponding calculations for other (still comparatively simple) finite graphs turn out to be quite complicated and hardly possible without the help of a computer.

In Connes' noncommutative geometry the commutator $[\mathfrak{D}, \hat{f}]$ represents a 'differential' df. The inequality which appears in the definition of the distance function can then be written as $\left\| \mathrm{d}f \right\| \leq 1$. Given a differential calculus (in the abstract algebraic sense), in order to have a distance function we need a definition of the norm of df. Connes defines it via a representation of the (first-order) differential algebra. In the case of discrete sets, it is natural to define a norm by

$$\left\| \mathrm{d}f \right\| = \sup\{ \left| f(k) - f(l) \right| / \rho_{kl}; (kl) \in E\} \tag{21}$$

where a digraph structure has been assigned to the set by the first-order differential calculus (see Dimakis and Müller-Hoissen, 1994, for details) and E denotes the set of its arrows. The positive constants ρ_{kl} assign an individual length to each arrow. The distance function is then taken to be

$$d(p, q) = \sup\{|f(p) - f(q)|; f \in \mathcal{A}, \|df\| \leq 1\} \tag{22}$$

This recipe reproduces the ordinary distances on the underlying graph.

Let $\mathcal{H} = \{\xi: E \cup \underline{E} \to \mathbb{C}\}$, where $\underline{E} = \{(kl) \in M \times M; (lk) \in E\}$ and M is the set of points of the digraph. Introducing operators

$$(\mathfrak{D}\xi)(kl) = \xi(lk)/\rho_{kl} \qquad (\forall \xi \in \mathcal{H}) \tag{23}$$

and

$$(\hat{f}\xi)(kl) = f(k)\xi(kl) \qquad (\forall \xi \in \mathcal{H}) \tag{24}$$

we get

$$([\mathfrak{D}, \hat{f}]\xi)(kl) = \frac{f(l) - f(k)}{\rho_{kl}} \xi(lk) \tag{25}$$

Demanding $\rho_{kl} = \rho_{lk}$, we find that this operator is given by a simple antisymmetric matrix with only a single entry per row and column. We obtain

$$\|[\mathfrak{D}, \hat{f}]\| = \|df\| \tag{26}$$

The last construction is adopted from the following result due to Rieffel (1993). Let $(\mathcal{M}, \rho)$ be a metric space. We choose $\mathcal{H} = l^2(\mathcal{M} \times \mathcal{M}\backslash\text{diagonal})$ as our Hilbert space and define

$$(\mathfrak{D}\xi)(x, y) = \xi(y, x)/\rho(x, y), \qquad (\hat{f}\xi)(x, y) = f(x)\xi(x, y) \tag{27}$$

The metric is then recovered from

$$d(x, y) = \sup\{|f(y) - f(x)|; \|f\|_L \leq 1\} \tag{28}$$

where $\|f\|_L = \sup\{|f(y) - f(x)|/\rho(x, y)\}$ is the Lipschitz number of f (Weaver, 1996).

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